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Networked gain-scheduled fault diagnosis under control input dropouts without data delivery acknowledgement

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SUMMARY

This paper investigates the fault diagnosis problem for discrete-time networked control systems under dropouts in both control and measurement channel with no delivery acknowledgment. We propose to use a proportional integral observer-based fault diagnoser collocated with the controller. The observer estimates the faults and computes a residual signal whose comparison with a threshold alarms the fault appearance. We employ the expected value of the arriving control input for the open loop estimation and the measurement reception scenario for the correction with a jump observer. The jumping gains are scheduled in real time with rational functions depending on a statistic of the difference between the control command being applied in the plant and the one being used in the observer. We design the observer, the residual and the threshold to maximize the sensitivity under faults while guaranteeing some minimum detectable faults under a predefined false alarm rate. Exploiting sum-of-squares decomposition techniques, the design procedure becomes an optimization problem over polynomials. Copyright © 2014 John Wiley & Sons, Ltd.

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KEY WORDS: Fault diagnosis, networked control systems, Markov jump systems, gain-scheduling, sum-of-squares.

1. INTRODUCTION

Control systems in industry are becoming more complex and communication networks enhance flexibility and ease of manoeuvre [1]. However, in networked control systems (NCS), where the elements of the control architecture are not collocated, these benefits are achieved at the expense of introducing some new issues as time delays and dropouts in the information transmission [2–5]. With the appearance of these network-induced problems, guaranteeing a reliable, safe and efficient operation of NCS has become a challenging concern in the last years, and researches have being adapting and improving traditional model-based fault diagnosis methods [6] to operate in networked environments [7–9].

Generally, when dealing with dropouts, the existing observer-based fault detection and estimation algorithms only consider measurement losses either by focusing on filter design or by assuming that the control input being applied at the plant is known when updating the observer [10–15]. Concerning measurements dropouts, the use of jump observers whose modes are related to the measurement transmission outcome improves estimation performances [16] (with respect to gain invariant approaches) and have been employed to fault detection in [10,12]. But, when the controller and fault diagnoser are collocated, and the controller to actuator link is offered by a network without successful delivery acknowledgement (motivated by reducing the network resource consumption,

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e.g. power and bandwidth), the above mentioned methods are not applicable and lead to poor performances.

Some works as [17] and [18] have addressed the problem of dealing with the control error induced by dropouts in the control channel, i.e., with the difference between the applied control input in the plant and the one used in the observer. The authors of [17] used the expected value of the control command at the plant to update the state estimation and designed the residual signal to guarantee some performances under faults and disturbances. After describing the involved residual signal in terms of the control error, [18] went one step beyond by generating a time-varying threshold adapted in real time to some control error statistics, allowing them to assure a predefined false alarm rate (FAR). However, in both works [17, 18] an observer-based residual generation schema with invariant gains was employed. Owing to guarantee robustness against all possible control errors, those approaches lead to conservative fault detection performances for control errors smaller than foreseen.

In the current work, we employ the expected value of the control input being applied at the plant to run the open loop fault estimation. We then derive a control error statistic available in real time that can be modelled by a bounded time-varying parameter. Based on [19, 20], the performance of a fault diagnosis algorithm can be defined by means of the trade-offs between the sensitivity to faults and the FAR. Seeking to improve fault diagnosis performances (e.g., time to detect faults), we introduce a gain-scheduled Markovian jump proportional integral observer to estimate the faults. The observer gain jumps with the measurement reception scenario, modelled as a Markov chain, and follows some function of the aforementioned control error statistic. We define the residual signal as a quadratic form of the estimated fault vector whose comparison with a threshold guarantees fault detection. The major novelty of this work lies in scheduling in real time the observer gains with the control error statistic.

We design the gain-scheduled jump proportional integral observer, the residual and the threshold in order to minimize the response time to faults, by minimizing the H_∞ norm from fault to fault estimation error subject to attain disturbance and measurement noise attenuation and to guarantee fault detection over some minimum detectable faults with a prescribed FAR for all the possible control error occurrences. To handle this optimization procedure we fix the gain-scheduling function to be polynomial and then, we exploit sum-of-squares (SOS) decomposition techniques (see [21,22]). Some previous works have applied SOS methods to nonlinear polynomial systems [23,24], to linear parametric varying (LPV) systems [25] or to quasi-LPV systems [26]. The conceptual novelty introduced with respect to those works is the employment of SOS methods to schedule the observer-based fault diagnoser with a time-varying control error statistic that depends on the behaviour of the network and is known in real time.

Notation : Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the real and positive real numbers set. Let A and B be some matrices. The maximum and minimum eigenvalues of A are denoted by $\overline{\lambda}(A)$ and $\underline{\lambda}(A)$ respectively. $A \preceq 0$ means that matrix A is negative semidefinite. Similar applies to $\prec$, $\succ$ and $\succeq$. The direct sum is represented as $\bigoplus$, where $A \bigoplus B$ is a block diagonal matrix with A and B on its diagonal. Let $x_k \in \mathbb{R}^n$ be a stochastic process. $\mathbf{E}\{\cdot\}$ and $\mathbf{Pr}\{\}$ denotes expectation and probability. We denote the RMS norm of process x_k by $\|x\|_{\mathrm{RMS}} = \lim_{K\to\infty} \sqrt{\frac{1}{K}\sum_{k=0}^{K-1} x_k^T x_k}$.

2. PROBLEM SETUP

In this work, we consider linear time-invariant discrete-time systems of the form

$$x_{k+1} = A\,x_k + B_u\,u_k + B_w\,w_k + B_f\,f_k, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, $w_k \in \mathbb{R}^{n_w}$ is the state disturbance, and $f_k \in \mathbb{R}^{n_f}$ is the fault vector. The measurable outputs of the system are defined by

$$y_k = C\,x_k, \tag{2}$$

where $y \in \mathbb{R}^{n_y}$ is the output vector.

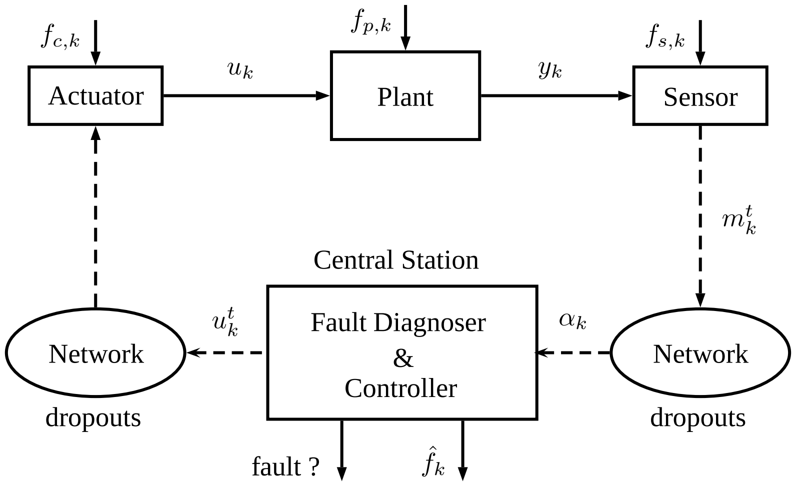

Figure 1. Networked fault diagnosis problem under dropouts with possible faults in the actuator (f_c), faulty components in the plant (f_p) and faulty sensor (f_s).

Each measurable output can be measured by at least one sensor (that may introduce faults), having $n_m \geq n_y$ sensors. We write the measured and transmitted value as

$$m_{s,k}^t = c_s\, x_k + h_s\, f_k + v_{s,k}, \quad s = 1, \dots, n_m \tag{3}$$

where $m_{s,k}^t \in \mathbb{R}$ is the k-th measurement of the s-th sensor and $v_{s,k} \in \mathbb{R}$ is the s-th sensor noise. c_s denotes a row of matrix C (different c_s can refer to the same row of C) and h_s each of the rows of matrix H. Both the state disturbance input and the measurement noise are assumed to be wide-sense stationary stochastic processes[†] with bounded variances where their RMS norms are bounded by $\|w\|_{\text{RMS}} \leq \bar{w}_{\text{rms}}$ and $\|v\|_{\text{RMS}} \leq \bar{v}_{\text{rms}}$ (with $v_k = [v_{1,k} \ \cdots \ v_{n_m,k}]^T$).

In the current work, we model the fault as a slow time-varying signal (cf. [27, 28]), i.e.,

$$f_{k+1} = f_k + \Delta f_k, \ \|\Delta f\|_\infty \leq \overline{\Delta f} \tag{4}$$

where Δf_k is the bounded fluctuation of the fault from instant k to $k+1$. This allows us to model, for instance, step signals (Δf_k would only be different from zero when the fault appears) or ramp signals (Δf_k has a constant value) that have been widely used to test fault detection algorithms [6, 29].

We aggregate the evolution of the system state (1) and the fault (4) leading to an extended order model defined by

$$z_{k+1} = \bar{A}\, z_k + \bar{B}_u\, u_k + \bar{B}_w\, w_k + \bar{B}_f\, \Delta f_k \tag{5}$$

with $z_k = \begin{bmatrix} x_k^T & f_k^T \end{bmatrix}^T$ and

$$\bar{A} = \begin{bmatrix} A & B_f \\ 0 & I \end{bmatrix}, \ \bar{B}_u = \begin{bmatrix} B_u \\ 0 \end{bmatrix}, \ \bar{B}_w = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \ \bar{B}_f = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where $z \in \mathbb{R}^{\bar{n}}$ with $\bar{n} = n + n_f$. Then, the measurements are

$$m_{s,k}^t = \bar{c}_s z_k + v_{s,k}, \quad s = 1, \dots, n_m \tag{6}$$

with $\bar{c}_s = [c_s \ h_s]$. We consider that the pair $(\bar{A}, \bar{C})$ is detectable (being $\bar{C}$ the matrix whose rows are $\bar{c}_s$), otherwise (i.e., $n_f > n_m$), only a combination of the faults can be detected.

[†]If x_k is wide sense stationary its RMS norm becomes $\|x\|_{\text{RMS}} = \sqrt{\mathbf{E}\{x_k^T x_k\}}$.

Remark 2.1
A transformation of the system when the pair $(\bar{A}, \bar{C})$ is undetectable must be carried out (leading to new $\bar{n}_f$ faults, a combination of the original faults, with $\bar{n}_f \leq n_m$) before the methods on the paper become valid, as proposed in [30].

In the current work, we consider that the fault diagnoser and the controller are collocated in a central station. We assume that sensors, central station and actuators communicate through a network without successful delivery acknowledgement of sent packets (e.g. UDP-like networks) where dropouts are likely to occur (see Figure 1). The control input sent to the actuators is assumed to be known.

2.1. Measurement reception modelling

Each sensor measures its output synchronously with the control input update and transmits, independently from each other, a time-tagged packet with the measurement $m^t_{s,k}$ to the central station through the network (see Figure 1). We model the reception state of each measurement from sensor $s = 1$ to n_m at instant k with

$$\alpha_{s,k} = \begin{cases} 1 & \text{if } m^t_{s,k} \text{ is acquired at instant } k, \\ 0 & \text{if } m^t_{s,k} \text{ is lost.} \end{cases} \tag{7}$$

We use $\alpha_k = \bigoplus_{s=1}^{n_m} \alpha_{s,k}$ to model the reception scenario at instant k of the whole transmitted measurements

$$m^t_k = \bar{C}\, z_k + v_k, \tag{8}$$

where $v_k = [v_{1,k} \; \cdots \; v_{n_m,k}]^T$ is the measurement noise vector. α_k is a diagonal matrix with binary variables in its diagonal elements. We assume that α_k is governed by a finite ergodic[‡] Markov chain [31] whose states are in the set

$$\alpha_k \in \Xi = \{\eta_0, \eta_1, \ldots, \eta_q\}, \;\; q = 2^{n_m} - 1, \tag{9}$$

where η_i (for $i = 0, \ldots, q$) represents each possible measurement reception scenario. η_0 denotes the case when $\alpha_k = 0$. The transition probability matrix $\Lambda = [p_{i,j}]$ with

$$p_{i,j} = \mathbf{Pr}\{\alpha_{k+1} = \eta_j \big| \alpha_k = \eta_i\}$$

is assumed to be known.

Remark 2.2
Assuming mutually independent Markovian processes for the packet dropouts (see [3, 5]), i.e.,

$$\begin{aligned}
&\mathbf{Pr}\{\alpha_{s,k} = 0 | \alpha_{s,k-1} = 0\} = q_s, \\
&\mathbf{Pr}\{\alpha_{s,k} = 1 | \alpha_{s,k-1} = 0\} = 1 - q_s, \\
&\mathbf{Pr}\{\alpha_{s,k} = 1 | \alpha_{s,k-1} = 1\} = p_s, \\
&\mathbf{Pr}\{\alpha_{s,k} = 0 | \alpha_{s,k-1} = 1\} = 1 - p_s,
\end{aligned}$$

for all $s = 1, \ldots, n_m$, each probability of matrix $\Lambda = [p_{i,j}]$ (for $i, j = 0, \ldots, q$) is computed as

$$p_{i,j} = \prod_{s=1}^{n_m} \mathbf{Pr}\{\alpha_{s,k} = \eta_{s,j} | \alpha_{s,k-1} = \eta_{s,i}\}$$

where $\eta_{s,i}$ is the s-th diagonal element of η_i.

‡In an ergodic Markov chain every state can be reached from every state in a finite time.

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2.2. Control input update modelling

At each instant $k-1$, the controller sends to the actuators (through the network) a single packet with all the control inputs to be used at instant k[§]. We denote by u_k^t the control input transmitted from the controller (at $k-1$) to be applied at instant k. We model the control input reception at instant $k-1$ with

$$\theta_{k-1} = \begin{cases} 1 & \text{if } u_k^t \text{ is received at instant } k-1, \\ 0 & \text{if } u_k^t \text{ is lost.} \end{cases} \tag{10}$$

Each actuator implements a zero order hold strategy, i.e.,

$$u_k = \begin{cases} u_k^t & \text{if } \theta_{k-1} = 1 \\ u_{k-1} & \text{otherwise.} \end{cases} \tag{11}$$

As the network involved in the communication has no acknowledgement of successful delivery, we ignore at the central station the exact value of the control input being applied at each instant. We assume that the probability of being applying at instant k the control input transmitted at $k-\tau-1$ is known, i.e.

$$\varphi_\tau = P\{u_k = u_{k-\tau}^t\}, \qquad \tau = 0, \ldots, N_u, \qquad \sum_{\tau=0}^{N_u} \varphi_\tau = 1, \tag{12}$$

where N_u denotes the maximum integer number of consecutive packet dropouts from the central station to the actuators.

Remark 2.3
Let us suppose that the dropouts in the controller to actuator link follow a Markovian process (see [3, 5]) with

$$\begin{aligned} \mathbf{Pr}\{\theta_k = 0 | \theta_{k-1} = 0\} &= q_u, \\ \mathbf{Pr}\{\theta_k = 1 | \theta_{k-1} = 0\} &= 1 - q_u, \\ \mathbf{Pr}\{\theta_k = 1 | \theta_{k-1} = 1\} &= p_u, \\ \mathbf{Pr}\{\theta_k = 0 | \theta_{k-1} = 1\} &= 1 - p_u. \end{aligned}$$

If the actuators implement a time-triggered protocol that force the controller to assure the successful control command transmission when the consecutive number of packet dropout is N_u, then the probabilities in (12) can be obtained as

$$\begin{aligned} \varphi_0 &= \frac{1}{p_{0,u}} \pi_{1,u}, \\ \varphi_{\tau>0} &= \frac{1}{p_{0,u}} q_u^{\tau-1} (1-p_u) \pi_{1,u}, \end{aligned}$$

where

$$p_{0,u} = \pi_{1,u} + \sum_{\tau=1}^{N_u} q_u^{\tau-1} (1-p_u) \pi_{1,u}$$

denotes the tail probability originated by the bounded number of consecutive dropouts N_u. $\pi_{1,u}$ is the probability of updating the control inputs (i.e., $\pi_{1,u} = \mathbf{Pr}\{\theta_k = 1\}$) that can be computed as $\pi_u = \pi_u \Lambda_u$, where $\pi_u = [\pi_{0,u}\ \pi_{1,u}]$ and Λ_u is the associated transition probability matrix of θ_k (being $\pi_{0,u} = \mathbf{Pr}\{\theta_k = 0\}$).

[§] This control strategy is used to overcome delays up to one instant, see [32].

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As the value of the real control input being applied to the system is unknown, we propose the use of its expected value $\mathbf{E}\{u_k\}$ to update the open loop observer estimation. Let us denote $\mathbf{E}\{u_k\}$ by u_k^c where

$$u_k^c = \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t. \tag{13}$$

With that definition, the control error $\tilde{u}_k = u_k - u_k^c$ (the difference between the control input being applied in the plant and the one being used in the observer) can be expressed as

$$\tilde{u}_k = u_k - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t. \tag{14}$$

The next lemma characterizes some statistics of $\tilde{u}_k$.

Lemma 2.1
The control error $\tilde{u}_k$ has the following properties:

$$\mathbf{E}\{\tilde{u}_k\} = 0, \tag{15a}$$

$$\mathbf{E}\{\tilde{u}_k^T \tilde{u}_k\} = \sum_{d=0}^{N_u} \varphi_d \left(u_{k-d}^t - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t \right)^T \left(u_{k-d}^t - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t \right). \tag{15b}$$

Proof
The expected value of $\tilde{u}_k$ is zero by definition of u_k^c, see (13). The expected value of $\tilde{u}_k^T \tilde{u}_k$ can be expressed as

$$\mathbf{E}\{\tilde{u}_k^T \tilde{u}_k\} = \mathbf{E}\left\{ \left(u_k - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t \right)^T \left(u_k - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t \right) \right\}$$
$$= \sum_{d=0}^{N_u} P\{u_k = u_{k-d}^t\} \, \mathbf{E}\left\{ \left(u_k - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t \right)^T \left(u_k - \sum_{d=0}^{N_u} \varphi_d \, u_{k-d}^t \right) \middle| u_k = u_{k-d}^t \right\}$$

where the total probability law has been applied, proving (15b). □

Let us use δ_k to denote $\mathbf{E}\{\tilde{u}_k^T \tilde{u}_k\}$. Note that the value of δ_k is known and can be calculated in real time with (15b) since the transmitted control input u_k^t is available at the central station. In the present work, we assume that δ_k is a bounded time-varying signal fulfilling $\delta_k \in \mathcal{S}$ with

$$\mathcal{S} = \{\delta_k \, : \, 0 \leq \delta_k \leq \bar{\delta}, \; \forall k\}. \tag{16}$$

Remark 2.4
The upper bound $\bar{\delta}$ can be calculated analysing the system and controller dynamic that may include magnitude saturation and rate limitation. However, the accurate calculation of this bound is not really necessary, since $\bar{\delta}$ only defines the search space of the optimization problem (that will be described later), and hence $\bar{\delta}$ may simply be selected to be a high enough value.

2.3. *Fault diagnosis algorithm*

Taking into account the previous analysis, the proposed fault estimation algorithm is as follows:

$$\hat{z}_{k^-} = \bar{A} \, \hat{z}_{k-1} + \bar{B}_u \, u_{k-1}^c, \tag{17a}$$
$$\hat{z}_k = \hat{z}_{k^-} + L_k \, \alpha_k \, (m_k^t - \bar{C} \, \hat{z}_{k^-}). \tag{17b}$$

At each instant k, we run the model in open loop using the expected value of the control input being applied at the plant u^c_{k-1}, see (17a). When some measurements are available at the central station, we update the estimation with the available information (nonzero diagonal elements of α_k) by means of the updating gain matrix L_k, see (17b). Otherwise, we hold the open loop estimation, i.e., $\hat{z}_k = \hat{z}_{k^-}$ (since $\alpha_k = 0$). Note that, taking into account the fault model defined in (4), the proposed estimation algorithm is in fact, a proportional integral observer (see [27, 28]).

Defining the extended state estimation error as $\tilde{z}_k = z_k - \hat{z}_k$, its dynamic is given by

$$\tilde{z}_k = \left(I - L_k\, \alpha_k\, \bar{C}\right) \left(\bar{A}\, \tilde{z}_{k-1} + B_{\mathcal{W}}\, \mathcal{W}_{k-1}\right) - L_k\, \alpha_k\, v_k \tag{18}$$

where $B_{\mathcal{W}} = [\bar{B}_w \;\; \bar{B}_f \;\; \bar{B}_u]$ and $\mathcal{W}_{k-1} = [w^T_{k-1} \;\; \Delta f^T_{k-1} \;\; \tilde{u}^T_{k-1}]^T$.

Using the estimated faults from (17b), as $\hat{f}_k = [0 \; I]\, \hat{z}_k$, we define the residual signal of the fault detection algorithm as

$$r_k = \hat{f}^T_k\, F^{-1}\, \hat{f}_k, \tag{19}$$

where F is some matrix to be defined. Then, the fault detection law is

$$\begin{cases} \text{if } r_k \le r^{\text{th}} & \text{no fault,} \\ \text{if } r_k > r^{\text{th}} & \text{fault,} \end{cases} \tag{20}$$

being r^{th} a constant threshold to be defined. Then, fault isolation is attained by combining fault estimation and fault detection, allowing us to identify the source of the faults.

Remark 2.5
Extending the definition of a minimum detectable fault given in [6], we define a minimum detectable fault as a fault that makes the residual cross a unitary threshold (i.e., $r^{\text{th}} = 1$), provided no other faults, disturbances, measurement noises and control errors are present. Then, under a zero fault estimation error (i.e. $\hat{f} = f$), if $r^{\text{th}} = 1$, the diagonal elements of F define the square of the minimum detectable faults, i.e., $F(l, l) = f^2_{\min,l}$ ($l = 1, \ldots, n_f$). To impose some given minimum detectable faults when the threshold r^{th} is chosen to be different from one, we must scale the residual generation in (19) by setting the diagonal elements of F as $F(l, l) = f^2_{\min,l}/r^{\text{th}}$. In this case, the only parameter value to be chosen is the threshold r^{th}, which is just a scaling factor. However if the residual generation (19) is already implemented, we can modify r^{th} in the residual evaluation (20) to change the FAR and the minimum detectable faults in a simple way.

Considering the fault detection decision law, a false alarm is produced if $r_k > r^{\text{th}}$ when $f_k = 0$ and the FAR is defined as the average probability of rising false alarms over an infinite-time window, i.e.

$$\Psi = \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \Pr\left\{ r_k > r^{\text{th}} \,\middle|\, f_k = 0 \right\}. \tag{21}$$

The aim of this work is to compute gain matrices L_k, matrix F, and threshold r^{th} such that the fault diagnoser attains disturbance and measurement noise attenuation, reaches some fault diagnosis performances, assures a given FAR and overcomes the uncertainty introduced by the control input dropouts for any $\delta_k \in \mathcal{S}$.

Using jump linear estimators that relate their jumps to the measurement reception improves the estimation performance with respect to employing invariant gain estimators [16, 33] and have been recently adapted to fault detection algorithms [10, 12]. When dealing with the uncertainty of the control input update, the authors of [18] propose to adapt the threshold to the mean and variance of the control error in order to improve the performance of their fault detector. However, they used an invariant gain observer approach that leads to conservative fault diagnosis performances for control errors smaller than anticipated. In the current work, in order to improve the fault diagnosis performance, we propose a Markovian jump estimator with a gain-scheduled approach depending on the real time values of δ_k, i.e.

$$L_k = L(\alpha_k, \Delta_{k-1}) = \begin{cases} L_i(\Delta_{k-1}) & \text{if } \alpha_k = \eta_i \text{ for } i = 1, \dots, q, \\ 0 & \text{if } \alpha_k = \eta_0. \end{cases} \tag{22}$$

where

$$\Delta_{k-1} = \begin{bmatrix} \delta_{k-1} & \delta_{k-2} \end{bmatrix}^T. \tag{23}$$

We schedule the updating gain L_k with both the current control error statistic δ_{k-1} (at instant k) and the past one δ_{k-2} (at instant $k-1$) to consider the present control uncertainty δ_{k-1} as well as the variation it has suffered from δ_{k-2} to δ_{k-1}. Note that, as the updating gain $L(\alpha_k, \Delta_{k-1})$ is scheduled with α_k and Δ_{k-1}, the residual signal r_k is also related to these parameters.

3. FAULT DIAGNOSER DESIGN

In this section we address the design of the gain-scheduled Markovian jump proportional integral observer with law $L(\alpha_k, \Delta_{k-1})$ as well as the residual r_k and its threshold r^{th} with an H_∞-based procedure. We first present a sufficient condition for the existence of such a fault diagnoser based on matrix inequalities that depend on the control error statistic Δ_k. This condition allows bounding the RMS norm of the fault estimation error vector. Second, we derive how to bound the FAR given by the fault diagnoser. Finally, we show that by restricting the dependences to be polynomial, we can solve the design problem in polynomial time through SOS methods.

The next theorem presents the H_∞ observer design based on a parameter-dependent matrix inequality.

Theorem 3.1
Consider the fault estimation algorithm (17a)-(17b) applied to system (1)-(4). If there exist positive definite symmetric matrices $P_j(\Delta_k) \in \mathbb{R}^{\bar{n}\times\bar{n}}$ and $F \in \mathbb{R}^{n_f\times n_f}$, full matrices $G_j(\Delta_k) \in \mathbb{R}^{\bar{n}\times\bar{n}}$ and $X_j(\Delta_k) \in \mathbb{R}^{\bar{n}\times n_m}$, and positive scalar functions $\gamma_w(\Delta_k) \in \mathbb{R}^+$, $\gamma_v(\Delta_k) \in \mathbb{R}^+$, $\gamma_u(\Delta_k) \in \mathbb{R}^+$ and $\gamma_f(\Delta_k) \in \mathbb{R}^+$ for all $i, j = 0, \dots, q$ and $\delta_k \in \mathcal{S}$ fulfilling

$$\Upsilon_i(\Delta_k, \Delta_{k-1}) = \begin{bmatrix} \Omega(\Delta_k) & \bar{M}_{A,i}(\Delta_k) & \bar{M}_{B,i}(\Delta_k) & 0 \\ \bar{M}_{A,i}(\Delta_k)^T & P_i(\Delta_{k-1}) & 0 & \bar{B}_f \\ \bar{M}_{B,i}(\Delta_k)^T & 0 & \Gamma(\Delta_k) & 0 \\ 0 & 0 & \bar{B}_f^T & F \end{bmatrix} \succ 0, \tag{24}$$

being Δ_k as in (23), with

$$\begin{aligned} \Omega(\Delta_k) &= \bigoplus_{j=0}^{q} G_j(\Delta_k) + G_j(\Delta_k)^T - P_j(\Delta_k), \\ \bar{M}_{A,i}(\Delta_k) &= \begin{bmatrix} \sqrt{p_{i,0}} M_{A,0}(\Delta_k)^T & \cdots & \sqrt{p_{i,q}} M_{A,q}(\Delta_k)^T \end{bmatrix}^T, \\ M_{A,j}(\Delta_k) &= (G_j(\Delta_k) - X_j(\Delta_k)\,\eta_j\,\bar{C})\,\bar{A}, \\ \bar{M}_{B,i}(\Delta_k) &= \begin{bmatrix} \sqrt{p_{i,0}} M_{B,0}(\Delta_k)^T & \cdots & \sqrt{p_{i,q}} M_{B,q}(\Delta_k)^T \end{bmatrix}^T, \\ M_{B,j}(\Delta_k) &= \begin{bmatrix} (G_j(\Delta_k) - X_j(\Delta_k)\,\eta_j\,\bar{C}) \begin{bmatrix} \bar{B}_w & \bar{B}_u & \bar{B}_f \end{bmatrix} & -X_j(\Delta_k)\eta_j \end{bmatrix}, \\ \Gamma(\Delta_k) &= \gamma_w(\Delta_k)\, I_{n_w} \oplus \gamma_u(\Delta_k)\, I_{n_u} \oplus \gamma_f(\Delta_k)\, I_{n_f} \oplus \gamma_v(\Delta_k)\, I_{n_m}, \end{aligned}$$

then, defining the observer gain matrices as $L_i(\Delta_k) = G_i(\Delta_k)^{-1} X_i(\Delta_k)$, the following statements are fulfilled for all $\alpha_k \in \Xi$, $\delta_k \in \mathcal{S}$, $\|w\|_{\text{RMS}} \leq \bar{w}_{\text{rms}}$, $\|v\|_{\text{RMS}} \leq \bar{v}_{\text{rms}}$ and $\|\Delta f\|_\infty \leq \overline{\Delta f}$:

i) In the absence of disturbances, faults, control errors and measurement noises, the extended state estimation error (18) converges asymptotically to zero in average.

ii) Under null initial conditions, the expected value of the squared RMS norm of the fault estimation error is bounded by

$$\mathbf{E}\{\|\tilde{f}\|^2_{\mathrm{RMS}}\} < \overline{\lambda}(F)\,\left(\bar{\gamma}_w \bar{w}^2_{\mathrm{rms}} + \bar{\gamma}_v \bar{v}^2_{\mathrm{rms}} + \bar{\gamma}_u + \bar{\gamma}_f\, n_f\, \overline{\Delta f}^2\right), \tag{25}$$

where

$$\bar{\gamma}_u = \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \gamma_u(\Delta_k)\,\delta_k, \quad \bar{\gamma}_\chi = \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \gamma_\chi(\Delta_k), \quad \chi = \{w, v, f\}. \tag{26}$$

Proof

If (24) holds, then we have $G_j(\Delta_k) + G_j(\Delta_k)^T - P_j(\Delta_k) \succ 0$ and thus, $G_j(\Delta_k)$ is a nonsingular matrix. As $P_j(\Delta_k)$ is a symmetric positive definite matrix, we can always state that

$$(\,P_j(\Delta_k) - G_j(\Delta_k))\;P_j(\Delta_k)^{-1}\,(\,P_j(\Delta_k) - G_j(\Delta_k))^T \succeq 0,$$

which implies that $G_j(\Delta_k) + G_j(\Delta_k)^T - P_j(\Delta_k) \preceq G_j(\Delta_k)\,P_j(\Delta_k)^{-1}\,G_j(\Delta_k)^T$. Substituting $X_j(\Delta_k)$ by $G_j(\Delta_k)\,L_j(\Delta_k)$, applying a congruence transformation on (24) by matrix

$$\left(\bigoplus_{j=0}^{q} G_j(\Delta_k)^{-1}\right) \oplus I \oplus I \oplus I,$$

taking Schur's complements and premultiplying the result by $\begin{bmatrix}\tilde{z}_k^T & w_k^T & \tilde{u}_k^T & \Delta f_k^T & v_{k+1}^T\end{bmatrix}$ and postmultiplying by its transpose leads to

$$\begin{aligned}&\sum_{j=0}^{q} p_{i,j}\,(\mathcal{A}_j(\Delta_k)\,\tilde{z}_k + \mathcal{B}_j(\Delta_k)\,\mathcal{W}_k)^T\,P_j(\Delta_k)\,(\mathcal{A}_j(\Delta_k)\,\tilde{z}_k + \mathcal{B}_j(\Delta_k)\,\mathcal{W}_k) - \tilde{z}_k^T\,P_i(\Delta_{k-1})\,\tilde{z}_k\\ &+ \tilde{f}_k^T\,F^{-1}\,\tilde{f}_k - \gamma_w(\Delta_k)\,w_k^T\,w_k - \gamma_u(\Delta_k)\,\tilde{u}_k^T\,\tilde{u}_k - \gamma_f(\Delta_k)\,\Delta f_k^T\,\Delta f_k - \gamma_v(\Delta_k)\;v_{k+1}^T\,v_{k+1} < 0,\end{aligned} \tag{27}$$

for all $i = 0, \ldots, q$ and $\delta_k \in \mathcal{S}$, where

$$\begin{aligned}\mathcal{A}_j(\Delta_k) &= (I - L_j(\Delta_k)\,\eta_j\,\bar{C})\bar{A},\\ \mathcal{B}_j(\Delta_k) &= \begin{bmatrix}(I - L_j(\Delta_k)\,\eta_j\,\bar{C})\begin{bmatrix}\bar{B}_w & \bar{B}_u & \bar{B}_f\end{bmatrix} & -L_j(\Delta_k)\,\eta_j\end{bmatrix},\\ \mathcal{W}_k &= \begin{bmatrix}w_k^T & \tilde{u}_k^T & \Delta f_k^T & v_{k+1}^T\end{bmatrix}^T.\end{aligned}$$

Now, let us define the Lyapunov function as $V_k = V(\tilde{z}_k, \alpha_k, \Delta_{k-1}) = \tilde{z}_k^T\,P_i(\Delta_{k-1})\,\tilde{z}_k$ for $\alpha_k = \eta_i$ and $i = 0, \ldots, q$.

i) In the absence of disturbances ($w_k = 0$), faults ($\Delta f_k = 0$), control errors ($\tilde{u}_k = 0$) and measurement noises ($v_k = 0$), expression (27) leads to

$$\tilde{z}_k^T\left(\sum_{j=0}^{q} p_{i,j}\mathcal{A}_j(\Delta_k)^T\,P_j(\Delta_k)\,\mathcal{A}_j(\Delta_k)\right)\tilde{z}_k - \tilde{z}_k^T\,P_i(\Delta_{k-1})\,\tilde{z}_k < 0 \tag{28}$$

for all $i = 0, \ldots, q$ and $\delta_k \in \mathcal{S}$, which assures $\mathbf{E}\{V_{k+1}|\alpha_k = \eta_i\} - V_k < 0$ guaranteeing that the extended state estimation error (18) converges asymptotically to zero in average for all the possible parameter values. This proves the first statement.

ii) For ease of notation let us write $\mathbf{E}\{V_{k+1}|\alpha_k = \eta_i\}$ as $\mathbf{E}\{V_{k+1}|\alpha_k\}$. Then, taking conditional expectation given α_{k-1} over expression (27), remembering that α_k is known at instant k, leads to

$$\begin{aligned}&\mathbf{E}\{V_{k+1}|\alpha_k\} - \mathbf{E}\{V_k\} + \mathbf{E}\{\tilde{f}_k^T\,F^{-1}\,\tilde{f}_k\} - \gamma_w(\Delta_k)\,\|w\|^2_{\mathrm{RMS}}\\ &\quad - \gamma_u(\Delta_k)\,\delta_k - \gamma_f(\Delta_k)\,\Delta f_k^T\,\Delta f_k - \gamma_v(\Delta_k)\,\|v\|^2_{\mathrm{RMS}} < 0,\end{aligned} \tag{29}$$

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for all $\alpha_k \in \Xi$ and $\delta_k \in \mathcal{S}$, where we have considered the assumptions on w_k and v_k, and that $\delta_k = \mathbf{E}\{\tilde{u}_k^T\, \tilde{u}_k\}$. For brevity, let us not include in the next the fact that the inequalities are fulfilled for all $\alpha_k \in \Xi$ and $\delta_k \in \mathcal{S}$. If (29) is fulfilled, then

$$\begin{aligned}&\mathbf{E}\{V_{k+1}|\alpha_k\} - \mathbf{E}\{V_k\} + \mathbf{E}\{\tilde{f}_k^T\, F^{-1}\, \tilde{f}_k\} - \gamma_w(\Delta_k)\, \bar{w}_{\rm rms}^2\\ &\quad - \gamma_u(\Delta_k)\, \delta_k - \gamma_f(\Delta_k)\, \Delta f_k^T\, \Delta f_k - \gamma_v(\Delta_k)\, \bar{v}_{\rm rms}^2 < 0\end{aligned} \tag{30}$$

holds because $\|w\|_{\rm RMS}^2 \leq \bar{w}_{\rm rms}^2$, $\|v\|_{\rm RMS}^2 \leq \bar{v}_{\rm rms}^2$. Under null initial conditions ($V_0 = 0$), adding the above expression from $k = 0$ to $K-1$ we obtain that

$$\begin{aligned}&\mathbf{E}\{V_K|\alpha_{K-1}\} + \sum_{k=0}^{K-1} (\mathbf{E}\{V_k|\alpha_{k-1}\} - \mathbf{E}\{V_k\}) + \sum_{k=0}^{K-1} \mathbf{E}\{\tilde{f}_k^T\, F^{-1}\, \tilde{f}_k\} - \sum_{k=0}^{K-1} \gamma_w(\Delta_k)\bar{w}_{\rm rms}^2\\ &- \sum_{k=0}^{K-1} \gamma_u(\Delta_k)\delta_k - \sum_{k=0}^{K-1} \gamma_f(\Delta_k)\, \Delta f_k^T\, \Delta f_k - \sum_{k=0}^{K-1} \gamma_v(\Delta_k)\bar{v}_{\rm rms}^2 < 0.\end{aligned} \tag{31}$$

Considering that $\mathbf{E}\{V_{K+1}|\alpha_K\} > 0$, dividing (31) by K and taking the limit when K tends to infinity we get

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{E}\{\tilde{f}_k^T\, F^{-1}\, \tilde{f}_k\} < \bar{\gamma}_w \bar{w}_{\rm rms}^2 + \bar{\gamma}_u + \bar{\gamma}_v \bar{v}_{\rm rms}^2 + \frac{1}{K} \sum_{k=0}^{K-1} \gamma_f(\Delta_k)\, \Delta f_k^T\, \Delta f_k \tag{32}$$

where we have taken into account the definition of $\bar{\gamma}_\_$ given in (26) and that

$$\lim_{K\to\infty} \frac{1}{K} \sum_{t=0}^{K-1} (\mathbf{E}\{V_k|\alpha_{k-1}\} - \mathbf{E}\{V_k\}) = 0.$$

Finally, due to the facts that

$$\begin{aligned}&\mathbf{E}\{\tilde{f}_k^T\, F^{-1}\, \tilde{f}_k\} \geq \underline{\lambda}(F^{-1})\, \mathbf{E}\{\tilde{f}_k^T\, \tilde{f}_k\},\\ &\gamma_f(\Delta_k)\, \Delta f_k^T\, \Delta f_k \leq \gamma_f(\Delta_k)\, n_f\, \|\Delta f\|_\infty^2 \leq \gamma_f(\Delta_k)\, n_f\, \overline{\Delta f}^2,\end{aligned}$$

and that $\underline{\lambda}(F^{-1}) = 1/\overline{\lambda}(F)$ (as F is a positive definite matrix), expression (32) leads to (25). □

Let us clarify and make some comments on the role of the decision matrix F and decision gains $\gamma_w(\Delta_k)$, $\gamma_u(\Delta_k)$, $\gamma_f(\Delta_k)$ and $\gamma_v(\Delta_k)$ in Theorem 3.1:

- Our approach uses updating gains $L(\alpha_{k+1}, \Delta_k)$ depending on the value of the control error statistic δ_k and δ_{k-1} . Then, gains $\gamma_\_(\Delta_k)$ (with $\gamma_\_ = \{\gamma_w,\, \gamma_u,\, \gamma_v,\, \gamma_f\}$) are also related to Δ_k to better characterize the propagation of the state disturbances, control errors, measurements noises and fault changes to the fault estimation error characterized in (25) for all $\delta_k \in \mathcal{S}$.
- If we fix F to assure some minimum detectable faults, we can extract from (25) that minimizing $\gamma_f(\Delta_k)$ increases the sensitivity of the fault diagnoser to faults (i.e, decreases the response time to faults) for all the possible $\delta_k \in \mathcal{S}_1$ (as it minimizes the upper bound of $\mathbf{E}\{\|\tilde{f}\|_{\rm RMS}^2\}$). If δ_k is time-invariant ($\delta_k = \delta_{k-1}$ for all k, and thus vector Δ_k has equal row values), then (25) and (26) lead to

$$\mathbf{E}\{\|\tilde{f}\|_{\rm RMS}^2\} < \overline{\lambda}(F)\, \left(\gamma_w(\delta_k)\bar{w}_{\rm rms}^2 + \gamma_v(\delta_k)\bar{v}_{\rm rms}^2 + \gamma_u(\delta_k)\delta_k + \gamma_f(\delta_k)\, n_f\, \overline{\Delta f}^2\right). \tag{33}$$

The next theorem extends the previous one showing how to bound the sensitivity of the fault diagnoser to state disturbances, measurement noises and control errors to bound the FAR given by the fault detection law (20).

Theorem 3.2
For a given threshold $r^{\text{th}} > 0$ and $0 \le \phi \le 1$, and under the premises of Theorem 3.1, if constraints (24) and

$$\Phi(\Delta_k) = \gamma_w(\Delta_k)\,\bar{w}_{\text{rms}}^2 + \gamma_v(\Delta_k)\,\bar{v}_{\text{rms}}^2 + \gamma_u(\Delta_k)\,\delta_k < \phi\, r^{\text{th}} \tag{34}$$

are fulfilled for all $\delta_k \in \mathcal{S}$, $\|w\|_{\text{RMS}} \le \bar{w}_{\text{rms}}$ and $\|v\|_{\text{RMS}} \le \bar{v}_{\text{rms}}$, then, the following additional statement holds:

iii) In the absence of faults and under null initial conditions, the fault detection logic (20) assures an upper bound of the FAR (21) given by ϕ.

Proof
Following the proof of Theorem 3.1, taking into account constraint (34), expression (32) leads to

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{E}\{\tilde{f}_k^T\, F^{-1}\, \tilde{f}_k\} < \phi\, r^{\text{th}} + \frac{1}{K}\sum_{k=0}^{K-1} \gamma_f(\Delta_k)\,\Delta f_k^T\,\Delta f_k. \tag{35}$$

In the absence of faults (i.e. $\tilde{f}_k = -\hat{f}_k$ and $\Delta f_k = 0$ for all k), we have $\mathbf{E}\{\tilde{f}_k^T\, F^{-1}\, \tilde{f}_k\} = \mathbf{E}\{\hat{f}_k^T\, F^{-1}\, \hat{f}_k\} = \mathbf{E}\{r_k\}$. Then, in the fault free case, (35) becomes

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{E}\{r_k | f_k = 0\} < \phi\, r^{\text{th}}. \tag{36}$$

Considering the above result and the FAR definition given in (21), we can employ Markov's inequality[¶] to obtain

$$\Psi = \lim_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \Pr\left\{r_k > r^{\text{th}} \,\middle|\, f_k = 0\right\} \le \lim_{K\to\infty} \frac{1}{K}\sum_{k=0}^{K-1} \frac{\mathbf{E}\{r_k | f_k = 0\}}{r^{\text{th}}} < \phi, \tag{37}$$

which ends this proof. □

Remark 3.1
In this work we propose a gain-scheduled fault diagnosis schema where the sensitivity to faults, through $\gamma_f(\Delta_k)$, is adapted to the control error (to improve the time response to faults) while the threshold r^{th} and the minimum detectable faults, defined by F, remain constant to guarantee the same minimum detectable faults over all the possible control errors. In the aim of [18], the presented method can be extended with not much effort to an adaptive threshold fault diagnosis procedure with a constant sensitivity to faults by imposing an invariant γ_f and a control error dependent matrix $F(\Delta_k)$. Then, the minimum detectable faults (which can also be seen as a part of the threshold, see (19) and (20)) would depend on the control error as proposed in [18] but with an observer that is also scheduled with the control error.

3.1. *SOS decomposition*

Conditions (24) and (34) in Theorem 3.2 lead to an infinite-dimensional problem. The main difficulty is how to verify the conditions over the entire parameter space. To deal with a finite-dimensional problem, we restrict the matrices and scalar functions in Theorem 3.2 to be polynomial functions of δ_k of fixed degree. Then, we can take advantage of SOS decompositions to turn the initial problem into a computationally tractable one.

Let us first show how to build polynomials of a given degree ([21, 22]).

[¶] If x is a positive random variable and $a > 0$, then $\mathbf{Pr}\{x > a\} \le \frac{\mathbf{E}\{x\}}{a}$.

Lemma 3.1
Let $x \in \mathbb{R}^n$ be some vector. $x^{\{d\}}$ is the vector of different monomials in x of degree not greater than d where its number is given by

$$\sigma(n,d) = \frac{(n+d)!}{n!\,d!}.$$

With that, full polynomial matrices $Q(x) \in \mathbb{R}^{N\times M}$ and polynomial symmetric matrices $P(x) \in \mathbb{R}^{N\times N}$ of degree $2d$ can be built with

$$Q(x) = Q\left(x^{\{2d\}} \otimes I_M\right), \tag{38a}$$

$$P(x) = \left(x^{\{d\}} \otimes I_N\right)^T P\left(x^{\{d\}} \otimes I_N\right), \tag{38b}$$

being matrix Q as $Q \in \mathbb{R}^{N\times M\sigma(n,2d)}$ and symmetric matrix P as $P \in \mathbb{R}^{N\sigma(n,d)\times N\sigma(n,d)}$.

Let us now characterize when a polynomial is said to be SOS ([21, 22]).

Lemma 3.2
Let $p(x)$ be a polynomial in $x \in \mathbb{R}^n$ of degree $2d$. Let $Z(x) \in \mathbb{R}^n$ be a vector with all the monomials in x of degree $\leq d$ as entries. Then, $p(x)$ is said to be SOS if and only if there is a positive semidefinite matrix Q fulfilling $p(x) = Z(x)^T QZ(x)$. The set of SOS polynomials in x is denoted by $\Sigma(x)$.

The next results are derived from the Positivstellensatz [21, 22] that states that polynomial feasibility conditions can be addressed by checking whether the condition is SOS.

Lemma 3.3
Let $p(x)$ be a polynomial in $x \in \mathbb{R}^n$, and let $X = \{x \in \mathbb{R}^n : g_j(x) \geq 0,\ j = 1,\ldots,m\}$. Suppose there exist SOS polynomials $s_j(x) \in \Sigma(x)$ $(j = 1,\ldots,m,\ x \in \mathbb{R}^n)$ fulfilling $p(x) - \sum_{j=1}^{m} s_j(x)\,g_j(x) \in \Sigma(x)$, then, the following condition holds: $p(x) \geq 0,\ \forall\, x \in X$.

Lemma 3.4
Let $P(x) \in \mathbb{R}^{N\times N}$ be a symmetric polynomial matrix in $x \in \mathbb{R}^n$ and let $X = \{x \in \mathbb{R}^n : g_j(x) \geq 0,\ j = 1,\ldots,m\}$. Suppose there exist SOS polynomials $s_j(x,v) \in \Sigma(x,v)$ $(j = 1,\ldots,m)$ fulfilling $v^T P(x)v - \sum_{j=1}^{m} s_j(x,v)\,g_j(x) \in \Sigma(x,v)$ with $v \in \mathbb{R}^N$, then, the following condition holds: $P(x) \succeq 0,\ \forall\, x \in X$.

The above lemma shows that verifying whether a polynomial function or a polynomial matrix is nonnegative over a domain defined by polynomial constraints can be formulated by means of sufficient LMI conditions. This kind of problems are handled by several LMI parsers as [34] and [35].

Making use of the previous lemmas, in the following theorem we present a sufficient condition to numerically find the parametric matrices and functions that guarantee the properties stated in Theorem 3.2. We use δ_1, δ_2, δ_3, $\Delta_1 = \begin{bmatrix}\delta_1 & \delta_2\end{bmatrix}^T$ and $\Delta_2 = \begin{bmatrix}\delta_2 & \delta_3\end{bmatrix}^T$ to denote independent SOS variables representing de possible values of δ_k, δ_{k-1}, δ_{k-2}, Δ_k and Δ_{k-1} respectively where $\delta_k \in \mathcal{S}$ for all k.

Theorem 3.3
For a given threshold $r^{\text{th}} > 0$ and $0 \leq \phi \leq 1$, if there exist a symmetric positive definite matrix $F \in \mathbb{R}^{n_f\times n_f}$, symmetric polynomial matrices

$$P_i(\Delta_1) = \left(\Delta_1^{\{d_P\}} \otimes I_{\bar{n}}\right)^T P_i\left(\Delta_1^{\{d_P\}} \otimes I_{\bar{n}}\right) \tag{39}$$

$$P_i(\Delta_2) = \left(\Delta_2^{\{d_P\}} \otimes I_{\bar{n}}\right)^T P_i\left(\Delta_2^{\{d_P\}} \otimes I_{\bar{n}}\right) \tag{40}$$

with symmetric matrices $P_i \in \mathbb{R}^{\bar{n}\sigma(2,d_P)\times\bar{n}\sigma(2,d_P)}$ for $i = 0,\ldots,q$, polynomial matrices

$$G_i(\Delta_1) = G_i\left(\Delta_1^{\{d_G\}} \otimes I_{\bar{n}}\right), \quad X_i(\Delta_1) = X_i\left(\Delta_1^{\{d_P\}} \otimes I_{n_m}\right), \tag{41}$$

with $G_i \in \mathbb{R}^{\bar{n}\times\bar{n}\sigma(2,d_G)}$ and $X_i \in \mathbb{R}^{\bar{n}\times n_m\sigma(2,d_X)}$ for $i = 0,\ldots,q$, and polynomial functions

$$\gamma_w(\Delta_1) = \gamma_w^T\,\Delta_1^{\{d_\gamma\}}, \quad \gamma_u(\Delta_1) = \gamma_u^T\,\Delta_1^{\{d_\gamma-1\}}, \quad \gamma_f(\Delta_1) = \gamma_f^T\,\Delta_1^{\{d_f\}}, \quad \gamma_v(\Delta_1) = \gamma_v^T\,\Delta_1^{\{d_\gamma\}} \tag{42}$$

with $\gamma_w \in \mathbb{R}^{\sigma(2,d_\gamma)}$, $\gamma_u \in \mathbb{R}^{\sigma(2,d_\gamma-1)}$, $\gamma_f \in \mathbb{R}^{\sigma(2,d_f)}$ and $\gamma_v \in \mathbb{R}^{\sigma(2,d_\gamma)}$, where $2d_P$, d_G, d_X, d_γ, $d_\gamma - 1$ and d_f are the degrees of the involved polynomials, that fulfill the following constraints

$$\mu^T\,P_i(\Delta_1)\,\mu - s_{P1,i}(\delta_1,\mu)\,h(\delta_1) - s_{P2,i}(\delta_2,\mu)\,h(\delta_2) \in \Sigma(\Delta_1,\mu), \tag{43a}$$
$$s_{P1,i}(\delta_1,\mu) \in \Sigma(\delta_1,\mu), \quad s_{P2,i}(\delta_2,\mu) \in \Sigma(\delta_2,\mu), \tag{43b}$$
$$\nu^T\,\Upsilon_i(\Delta_1,\Delta_2)\,\nu - s_{\Upsilon1,i}(\delta_1,\nu)\,h(\delta_1) - s_{\Upsilon2,i}(\delta_2,\nu)\,h(\delta_2) - s_{\Upsilon3,i}(\delta_3,\nu)\,h(\delta_3) \in \Sigma(\Delta_1,\Delta_2,\nu) \tag{43c}$$
$$s_{\Upsilon1,i}(\delta_1,\nu) \in \Sigma(\delta_1,\nu), \quad s_{\Upsilon2,i}(\delta_2,\nu) \in \Sigma(\delta_2,\nu), \quad s_{\Upsilon3,i}(\delta_3,\nu) \in \Sigma(\delta_3,\nu), \tag{43d}$$
$$\left(\phi\,r^{\mathrm{th}} - \Phi(\Delta_k)\right) - s_{r1}(\delta_1)\,h(\delta_1) - s_{r2}(\delta_2)\,h(\delta_2) \in \Sigma(\Delta_1), \tag{43e}$$
$$s_{r1}(\delta_1) \in \Sigma(\delta_1), \quad s_{r2}(\delta_2) \in \Sigma(\delta_2), \tag{43f}$$
$$\gamma_j(\Delta_1) - s_{j1}(\delta_1)\,h(\delta_1) - s_{j2}(\delta_2)\,h(\delta_2) \in \Sigma(\Delta_1), \tag{43g}$$
$$s_{j1}(\delta_1) \in \Sigma(\delta_1), \quad s_{j2}(\delta_2) \in \Sigma(\delta_2), \quad j = \{w,\,u,\,f,\,v\}, \tag{43h}$$

for $i = 0,\ldots,q$ with $\Upsilon_i(\cdot)$ as in (24), $\Phi(\cdot)$ as in (34), $\mu \in \mathbb{R}^{\bar{n}}$, $\nu \in \mathbb{R}^{n_\nu}$ (with $n_\nu = \bar{n}\,(q+2) + 2\,n_f + n_w + n_u + n_m$) and

$$h(\delta) = \delta\,(\bar{\delta} - \delta),, \tag{44}$$

then constraints of Theorem 3.2 hold.

Proof
First, note that the set $\mathcal{S}$ is rewritten with its corresponding polynomial h as $\mathcal{S} = \{\delta\,:\,h(\delta) \geq 0\}$, see (44). Independent SOS variables δ_1, δ_2, δ_3 must fulfill $h(\cdot) \geq 0$ as $\delta_k \in \mathcal{S}$. Second, by Lemma 3.3 and Lemma 3.4 constraints (43c) and (43d) assure the positive definiteness of $\Upsilon_i(\Delta_1,\Delta_2)$ for any $\delta_1,\delta_2,\delta_3 \in \mathcal{S}$, which guarantee (24) in Theorem 3.1. Third, by Lemma 3.3 constraints in (43e) and (43f) asserts (34) in Theorem 3.2 for any $\delta_1,\delta_2 \in \mathcal{S}$. Finally, by Lemma 3.3 and Lemma 3.4 it follows that by constraints (43a), (43b), (43g) and (43h), we guarantee the positive definiteness of $P_i(\Delta_1)$, $\gamma_w(\Delta_1)$, $\gamma_u(\Delta_1)$, $\gamma_f(\Delta_1)$ and $\gamma_v(\Delta_1)$ for any $\delta_1,\delta_2 \in \mathcal{S}$ as required in Theorem 3.1. $\square$

In the above feasibility SOS problem μ and ν are scalarization vectors used to transform polynomial matrices in polynomials (see Lemma 3.4). The decision variables are matrices P_i, G_i, X_i, F, γ_w, γ_u, γ_f and γ_v; and also the coefficients of SOS polynomials $s_{\_}$ in (43). We propose to choose the degree of these SOS polynomials in such a way that all the addends in each SOS expression have equal degree on all the variables. This can be performed by choosing d_P, d_G, d_X, d_γ and d_f, and then setting for all $j = 1,2$ and $i = 0,\ldots,q$

$$\deg\ s_{Pj,i}(\delta_j,\mu) = \deg\ \left\{\delta_j^{\max\{2d_P-2,\,0\}},\ \mu^2\right\}, \tag{45a}$$
$$\deg\ s_{\Upsilon j,i}(\delta_j,\nu) = \deg\ \left\{\delta^{\max\{2d_P-2,\,d_G-2,\,d_X-2,0\}},\ \nu^2\right\}, \tag{45b}$$
$$\deg\ s_{\Upsilon3,i}(\delta_3,\nu) = \deg\ \left\{\delta^{\max\{2d_P-2,0\}},\ \nu^2\right\}, \tag{45c}$$
$$\deg\ s_{rj}(\delta_j) = \deg\ \delta_j^{\max\{d_\gamma-2,\,0\}}, \tag{45d}$$
$$\deg\ s_{wj}(\delta_j) = \deg\ s_{vj}(\delta_j) = \deg\ \delta_j^{\max\{d_\gamma-2,\,0\}}, \tag{45e}$$
$$\deg\ s_{uj}(\delta_j) = \deg\ \delta_j^{\max\{d_\gamma-3,\,0\}}, \tag{45f}$$
$$\deg\ s_{fj}(\delta_j) = \deg\ \delta_j^{\max\{d_f-2,\,0\}}, \tag{45g}$$

where $\deg$ returns the maximum degree for each variable in the involved polynomial.

Remark 3.2
We have set the degree of polynomial $\gamma_u(\Delta_1)$ to be $d_\gamma - 1$ in order to assure the same degree on δ_k in the addends of expression (34).

Remark 3.3
The polynomial degrees defined by d_P, d_G, d_X, d_γ and d_f can be seen as trade-off parameters between conservativeness and computational effort. Moreover, if $d_G = 0$ and $d_X = 0$, the resulting updating gain L does not depend on Δ_k. If $d_G = 0$ and $d_X > 0$, matrix L has a polynomial form on Δ_k while if $d_G > 0$ and $d_X \geq 0$ it has a rational expression. Therefore, d_G and d_X define the dependency of L on Δ_k.

Remark 3.4
The gain-scheduling implementation is as follows. First, knowing the sent control inputs u_k^t, at each instant k we compute the corresponding control error given by δ_{k-1} from expression (15b). Second, with the current value δ_{k-1}, the previous calculated one δ_{k-2} and the present measurement reception scenario $\alpha_k = \eta_i$ we calculate the observer gain L_k as $G_i(\Delta_{k-1})^{-1}X_i(\Delta_{k-1})$ where $G_i(\Delta_{k-1})$ and $X_i(\Delta_{k-1})$ are polynomial matrices given by (41) in Theorem 3.3, and Δ_{k-1} is the vector defined in (23).

3.2. *Fault diagnosis design strategy*

Taking into consideration the fault diagnosis performances derived from Theorem 3.1 and Theorem 3.2, we propose the following strategy based on an optimization problem to design the fault diagnoser.

We force the fault diagnoser to only detect faults beyond some values (avoiding to rise alarms when faults are small), i.e., we impose each of the minimum detectable faults $f_{\min,l}$ (for $l = 1, \ldots, n_f$), while we intend to detect the presence of faults as fast as possible (higher fault sensitivity under faults) with a guaranteed FAR. We address the fulfilment of these constraints in the next optimization procedure.

Optimization problem 1 (OP1)
For a unitary threshold $r^{\text{th}} = 1$ and a given desired FAR ψ, fix ϕ to be $\phi = \psi$. Let F be a diagonal matrix such as $F = \bigoplus_{l=1}^{n_f} f_{\min,l}^2$ where $f_{\min,l}$ stands for the minimum detectable fault. Then, the minimization problem

$$\begin{aligned} \text{minimize} \quad & J \\ \text{subject to} \quad & (43), \\ & J - \gamma_f(\Delta_1) - s_{J1}(\delta_1)\, h(\delta_1) - s_{J2}(\delta_2)\, h(\delta_2) \in \Sigma(\Delta_1), \\ & s_{J1}(\delta_1) \in \Sigma(\delta_1), \quad s_{J2}(\delta_2) \in \Sigma(\delta_2) \end{aligned} \tag{46}$$

with $\deg\ s_{J1}(\delta_1) = \deg\ \delta_1^{\max\{d_f-2,0\}}$ and $\deg\ s_{J2}(\delta_2) = \deg\ \delta_2^{\max\{d_f-2,0\}}$, assures fault detection over $f_{\min,l}$ (with $l = 1, \ldots, n_f$) with a FAR below ψ and maximizes in the worst case the sensitivity to faults of the resulting fault diagnoser for any possible δ_k .

Including constraints (43) in the above optimization problem guarantees the results of Theorem 3.2 as stated in Theorem 3.3. The new constraint imposes that

$$\gamma_f(\Delta_1) < J, \quad \forall \Delta_1 \in \{\delta_1, \delta_2 \,:\, h(\delta_1) \geq 0,\ h(\delta_2) \geq 0\},$$

which means that J is an upper bound of $\gamma_f(\Delta_k)$ for any k with $\delta_k \in \mathcal{S}$ (worst case). Then, $\bar{\gamma}_f < J$ (see (26)) and by minimizing J we minimize the upper bound of $\mathbf{E}\{\|\tilde{f}\|_{\text{RMS}}^2\}$ (see (25)) in the worst case.

Remark 3.5
Setting zero degree polynomials, (46) can be rewritten as a LMI optimization problem (worst case

LMI design) such as

$$\begin{array}{ll} \text{minimize} & J \\ \text{subject to} & \Upsilon_i \succ 0, \\ & \gamma_w\,\bar{w}_{\text{rms}}^2 + \gamma_v\,\bar{v}_{\text{rms}}^2 + \gamma_u\,\bar{\delta} < 1, \\ & \gamma_f < J \end{array} \tag{47}$$

for all $i = 0, \ldots, q$ with Υ_i as in (24).

The previous optimization (46) may lead to quite conservative results because $\gamma_f(\Delta_1)$ is minimized for the worst case. In order to obtain less conservative results, we propose as an alternative to introduce some weighting function $g(\Delta_1)$ over $\gamma_f(\Delta_1)$ such that

$$\Gamma_f = \int_{\mathcal{S}\times\mathcal{S}} g(\Delta_1)\,\gamma_f(\Delta_1)\,\mathrm{d}\Delta_1. \tag{48}$$

The next optimization problem shows how to modify OP1 to include the weighting function $g(\Delta_1)$.

Optimization problem 2 (OP2)
Consider some weighting function $g(\Delta_1)$, for a unitary threshold $r^{\text{th}} = 1$ and a given desired FAR Ψ, fix ϕ to be $\phi = \psi$. Let F be a diagonal matrix such as $F = \bigoplus_{l=1}^{n_f} f_{\min,l}^2$ where $f_{\min,l}$ stands for the minimum detectable fault. Then, the minimization problem

$$\begin{array}{ll} \text{minimize} & J \\ \text{subject to} & (43), \\ & \Gamma_f < J, \end{array} \tag{49}$$

where Γ_f is defined in (48), assures fault detection over $f_{\min,l}$ (with $l = 1, \ldots, n_f$) with a FAR below ψ and maximizes the sensitivity to faults of the resulting fault diagnoser under the weighting function $g(\Delta_1)$.

Remark 3.6
An interesting choice for function $g(\Delta_1)$ is the one that represents the case when δ_k is time invariant (constant or ramp-like transmitted control inputs, see (15b)), that is $\delta_1 = \delta_2$, and all the possible values of $0 \le \delta_k \le \bar{\delta}$ are equally weighted (i.e., assuming no knowledge about which value is more likely). This weighting function can be defined as

$$g(\Delta_1) = \begin{cases} \dfrac{1}{\bar{\delta}} & \text{if } \delta_2 = \delta_1 \text{ and } 0 \le \delta_1 \le \bar{\delta}, \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

4. EXAMPLES

4.1. Example 1

For ease of analysis, let us first consider a simple linear time invariant discrete time system defined by (1)-(3) with

$$A = \begin{bmatrix} 0.48 & 0.11 \\ 0.11 & 0.97 \end{bmatrix}, \quad B_u = \begin{bmatrix} -0.5 \\ 0.7 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix}, \quad C = \begin{bmatrix} 0.18 & 0.8 \end{bmatrix},$$

where the measurable output is measured by just one sensor.

The state disturbances and measurement noises are Gaussian noises with zero mean and bounded RMS norms given by

$$\|w\|_{\text{RMS}} = 0.05, \quad \|v\|_{\text{RMS}} = 0.01.$$

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The dropouts on the sensor to central station link follow a Markov chain with

$$\mathbf{Pr}\{\alpha_{1,k}=0|\alpha_{1,k-1}=0\}=0.4, \quad \mathbf{Pr}\{\alpha_{1,k}=1|\alpha_{1,k-1}=1\}=0.7.$$

The transition probability matrix of the possible measurement transmission outcome is given by

$$\Lambda=\begin{bmatrix}0.4 & 0.6\\ 0.3 & 0.7\end{bmatrix}.$$

The dropouts in the central station to controller link follow a modified version of the above Markov chain with a maximum number of consecutive dropouts of $N_u=6$ (see Remark 2.3). Then the probabilities of being using the control input sent $\tau-1$ instants before are

$$\varphi=\begin{bmatrix}0.668 & 0.2 & 0.08 & 0.032 & 0.0128 & 0.005 & 0.002\end{bmatrix},$$

where $\varphi=[\varphi_0\ \varphi_1\ \varphi_2\ \varphi_3\ \varphi_4\ \varphi_5\ \varphi_6]$. Let us assume that in the worst case the control error statistic δ_k is bounded by $0\leq\delta_k\leq 0.1$ for all k, i.e. $\bar{\delta}=0.1$.

We address the problem of detecting possible faults from the actuator over 1.2 with a FAR under 0.1, which is the most interesting case due to the control input dropout (without delivery acknowledgement), i.e.

$$B_f=\begin{bmatrix}-0.5\\ 0.7\end{bmatrix}, \quad H=0, \quad F=f_{\min}^2=1.2^2, \quad \phi=0.1.$$

Let us first analyse the differences between using observer constant gains via the optimization problem in Remark 3.5 (worst case LMI design) and scheduling ones with second degree polynomials via design procedure OP1 (worst case SOS design) and OP2 with $g(\Delta_1)$ as defined in (50), see cases C1, C2 and C3 in Table I (all with $r^{\text{th}}=1$). We represent the gain $L(\alpha,\Delta_1)$ as $L(\alpha,\Delta_1)=[L_{x_1}(\alpha,\Delta_1)\ L_{x_2}(\alpha,\Delta_1)\ L_f(\alpha,\Delta_1)]^T$ (we discard the analysis on $L_{x_2}(\alpha,\Delta_1)$ as its variation is not significant). Let us remember that $\Delta_1=\begin{bmatrix}\delta_1 & \delta_2\end{bmatrix}^T$, where δ_1 models the current control error statistic and δ_2 de previous one. Figure 2 illustrates, when $\delta_2=\delta_1$ (i.e., δ_k time invariant), how the observer gains and the value of $\gamma_f(\Delta_1)|_{\delta_2=\delta_1}$ adapt their value to δ_1 when a measurement is received (i.e., $L(\alpha=1,\,\Delta_1)|_{\delta_2=\delta_1}$). The values displayed in Figure 2 have been normalized dividing them by the results from C1. Design C1 (constant gains with respect to Δ_1) is the most conservative one as it keeps γ_f constant at the highest value for all δ_1. C2 improves the performance of C1 reducing γ_f up to a 5% (which increases the sensitivity under faults) when $\delta_1=0$ thanks to employing a scheduling observer gain. Note that the index value J in Table I is the same for C1 and C2 since their design methods focus on the minimization of the upperbound of $\gamma_f(\Delta_1)$ (for any Δ_1). Design C3 requires selecting the weighting function $g(\Delta_1)$, but even with the simple choice proposed in Remark 3.6, it leads to the least conservative results, dividing by 10 the value of γ_f when $\delta_1=0$ with respect to its maximum value (which also leads to the minimum value of index J in Table I). Figure 2 also shows that the proposed polynomial methods allow to obtain more sensitive fault diagnoser to faults (improving fault detection and estimation response) with respect to the constant gain observer for all the possible values of δ_k, whenever δ_k is smaller than its upper bound $\bar{\delta}$. From now on, we will only focus on C1 and C3.

Table I. Analysed cases in Example 4.1.

Case	OP	d_P	d_G	d_X	d_γ	d_f	J
C1	1	0	0	0	0	0	154
C2	1	1	2	2	2	2	154
C3	2	1	2	2	2	2	63

Let us now examine for case C3 the full behaviour of $L_f(\alpha=1,\Delta_1)$ for any $\delta_1,\delta_2\in\mathcal{S}$ normalized with respect to the correspondent gain of C1. Figure 3 shows that the updating gain

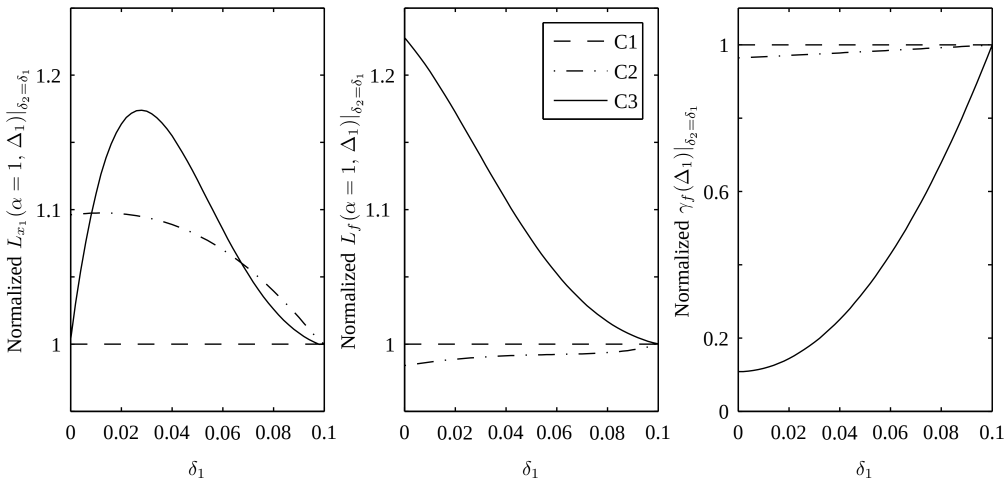

Figure 2. Comparison of the results given by optimization procedures C1, C2 and C3 normalized with respect to outcomes from C1 (see Table I) when δ_k is time-invariant.

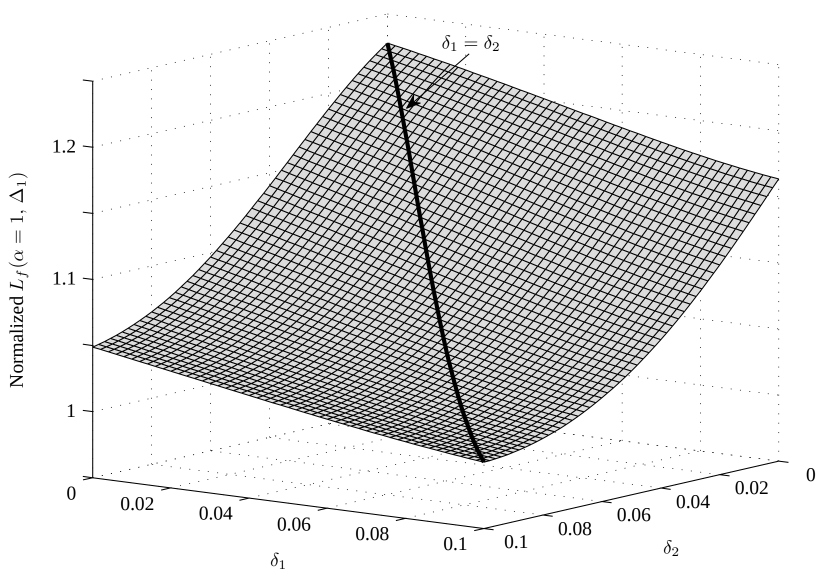

Figure 3. Normalized scheduled observer gain for C3 with respect to the one resulting from C1 in Table I.

value becomes smaller as the control error increases. The maximum value of the gain occurs when there is no control error $\delta_1 = \delta_2 = 0$, while the minimum corresponds to the case when the control error is maximum $(\delta_1 = \delta_2 = \bar{\delta})$.

Finally, let us evaluate the behaviour in simulation of the fault diagnosers for C1 and C3 under the appearance of step faults a 25% higher than $f_{\min}$ at $k = 10$ and $k = 60$ (vanishing at $k = 30$ and $k = 80$ respectively). The observer gain for C1 when $\alpha_k = 1$ is given by a constant matrix $L_{C1}(\alpha_k = 1) = \begin{bmatrix} -0.42 & 1.35 & 0.49 \end{bmatrix}^T$, while for C3 is a gain-scheduled matrix whose scheduling

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law is given by $L_{C3}(\alpha_k = 1, \Delta_{k-1}) = G_{C3}(\alpha_k = 1, \Delta_{k-1})^{-1} X_{C3}(\alpha_k = 1, \Delta_{k-1})$ where

$$
\begin{aligned}
G_{C3}(\alpha_k = 1, \Delta_{k-1}) = &\begin{bmatrix} 18.3 & 9.6 & -6.8 \\ 5.5 & 13.2 & -10.8 \\ 5 & -7.8 & 26.7 \end{bmatrix} + \begin{bmatrix} 53 & -268.4 & 810.7 \\ -8 & -73.8 & 195.3 \\ 33.6 & -28.3 & 118.8 \end{bmatrix} \delta_{k-1} \\
&+ \begin{bmatrix} -37.6 & -24.5 & 32.9 \\ 5.6 & -3.8 & -7.3 \\ -44.3 & -29.4 & 24.6 \end{bmatrix} \delta_{k-2} + \begin{bmatrix} -362.6 & 2865.7 & -8281.4 \\ 297.5 & 875.6 & -2110.7 \\ -533.4 & 225.7 & -1139.3 \end{bmatrix} \delta_{k-1}^2 \\
&\begin{bmatrix} 17.2 & 105.2 & -424.4 \\ -178.7 & -75.5 & -99.5 \\ 134.3 & 28.9 & 37.6 \end{bmatrix} \delta_{k-1}\delta_{k-2} + \begin{bmatrix} -89.2 & 186.4 & -603.1 \\ 23.3 & 91.4 & -133.7 \\ -111.2 & 1.6 & -57.3 \end{bmatrix} \delta_{k-2}^2, \\
X_{C3}(\alpha_k = 1, \Delta_{k-1}) = &\begin{bmatrix} 1 \\ 8.8 \\ 3.6 \end{bmatrix} + \begin{bmatrix} 4.3 \\ 16.7 \\ -48.9 \end{bmatrix} \delta_{k-1} + \begin{bmatrix} 16 \\ -10.6 \\ -5.4 \end{bmatrix} \delta_{k-2} + \begin{bmatrix} -23.8 \\ -85.9 \\ 244.6 \end{bmatrix} \delta_{k-1}^2 \\
&+ \begin{bmatrix} -105.7 \\ -30.9 \\ -42.6 \end{bmatrix} \delta_{k-1}\delta_{k-2} + \begin{bmatrix} 5.4 \\ 59.3 \\ 2.2 \end{bmatrix} \delta_{k-2}^2
\end{aligned}
$$

and δ_{k-1} and δ_{k-2} are the scheduling parameters computed in real time through (15b). Figure 5 presents the involved control inputs in the networked fault diagnosis schema u_k, u_k^t and u_k^c, as well as the evolution in time of the squared control error $\tilde{u}_k^2$ and its statistic δ_k. We appreciate that while $\tilde{u}_k^2$ may have abrupt changes, δ_k changes slowly and smoothly. Figure 4 shows the fault estimation and fault detection performances of the analysed situation. The first fault appears when δ_k is near 0 and almost constant. Then, thanks to its scheduled updating gain the fault diagnoser for C3 reduces the fault detection time in a 40% and the fault estimation settling time (at the 95%) in a 50% with respect to C1. When the second fault occurs δ_k is more time-dependent and its value is near 0.07. In these conditions both fault diagnosers have the same performances. This proves that using a gain scheduling approach allows to improve the fault detection and estimation performances when δ_k is small, while for high δ_k values (near its maximum), our method retrieves the performances of the H_∞ constant gain design.

4.2. *Example 2*

Let us now consider a more complex system by extending the previous one with

$$
B_u = \begin{bmatrix} -0.5 & -0.3 \\ 0.7 & 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.18 & 0.8 \\ 0 & 1 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.5 & 0 \\ 0.7 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
$$

In this example, the system has two measured outputs, two control inputs and we desire to diagnose faults from the first actuator and first sensor. Successful transmissions from the new sensor to the central unit follow a Markov chain with

$$
\mathbf{Pr}\{\alpha_{2,k} = 0 | \alpha_{2,k-1} = 0\} = 0.3, \quad \mathbf{Pr}\{\alpha_{2,k} = 1 | \alpha_{2,k-1} = 1\} = 0.5,
$$

which is different and independent of the successful transmission from the existing sensor.

Similar than in Example 4.1 we aim to diagnose faults over $f_{\min} = 1.2$ (for both channels, $F = 1.2^2 \oplus 1.2^2$) with a FAR under 0.1 ($\phi = 0.1$). However, we now examine the obtained performances resulting from adapting the observer gain to the measurement reception scenarios α_k in addition to scheduling it with Δ_k, see cases C4 and C5 in Table II.

Figure 6 shows the estimation and detection performances of the fault diagnosers from C4 and C5 under the appearance of two faults given by

$$
f_{1,k} = \begin{cases} 0, & \text{if } 0 \le k < 10 \\ 2\sin\left(\frac{k-10}{8}\right), & \text{if } 10 \le k < 85 \\ 0, & \text{if } k \ge 85 \end{cases}, \quad f_{2,k} = \begin{cases} 0, & \text{if } 0 \le k < 20 \\ 3\exp\left(-\frac{k-20}{10}\right), & \text{if } 20 \le k < 85 \\ 0, & \text{if } k \ge 85 \end{cases}
$$

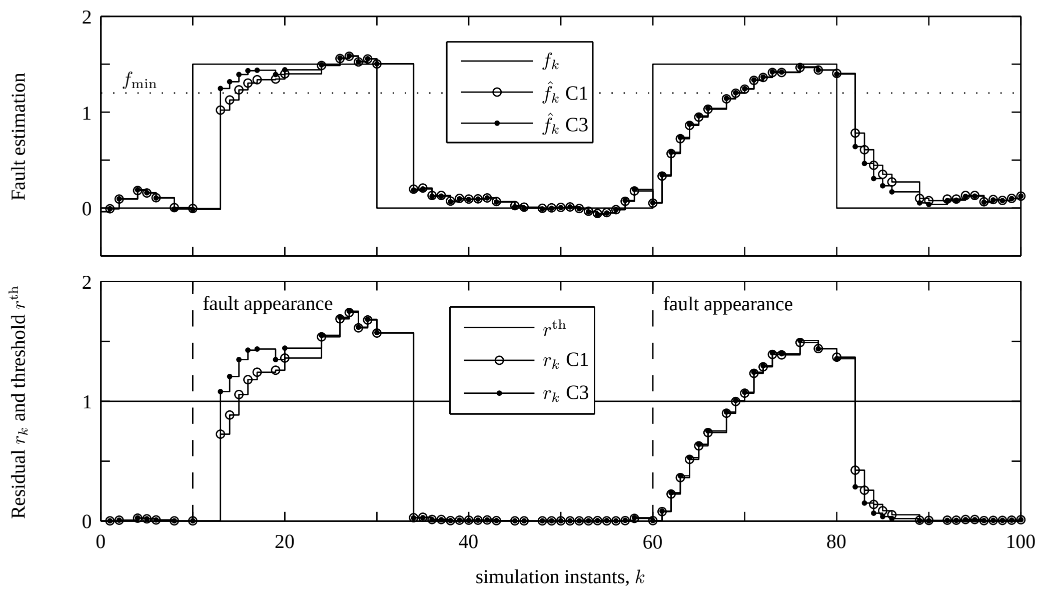

Figure 4. Fault estimation and fault detection performances in simulation for cases C1 and C3 in Table I. Markers • and ∘ denote instants when a measurement is received.

Table II. Analysed cases in Example 4.2.

Case	Obsv. gain	OP	d_P	d_G	d_X	d_γ	d_f	J
C4	$L(\alpha_k, \Delta_{k-1})$	2	1	2	2	2	2	42
C5	$L(\Delta_{k-1})$	2	1	2	2	2	2	104

From Table II and Figure 6 we appreciate that using an observer gain that jumps with the measurement reception scenarios (C4) improves the fault estimation and detection performances if compared to employing an observer gain that do not depend on α_k. This is achieved at the expense of storing a number of gain matrices that is three times higher than the number for C5. For the analysed case in Figure 6, estimator C4 reduces the RMS norm of the fault estimation error in simulation by a 23% with respect to C5.

5. CONCLUSION

In the present work, we designed a proportional integral observer-based fault diagnoser to operate under control input and measurement dropouts without successful delivery acknowledgement.

We modelled the measurement reception scenario as a Markov chain and characterized the control error between the control input being applied in the plant and the one being used in the observer. With that, the observer gain is related to the measurement transmission outcome and is scheduled in real time with a rational function of a control error statistic. We generated the residual signal using a quadratic form of the estimated faults provided by the observer, whose comparison to a threshold leads to fault detection.

The proposed design method allows minimizing the response time under faults while guaranteeing fault detection over some minimum detectable faults for a prescribed false alarm rate. We showed that our gain scheduling approach retrieves the performance of the worst case design with constant gains when the scheduling parameter is near to its upper bound, but improves it whenever the control error statistic is lower.

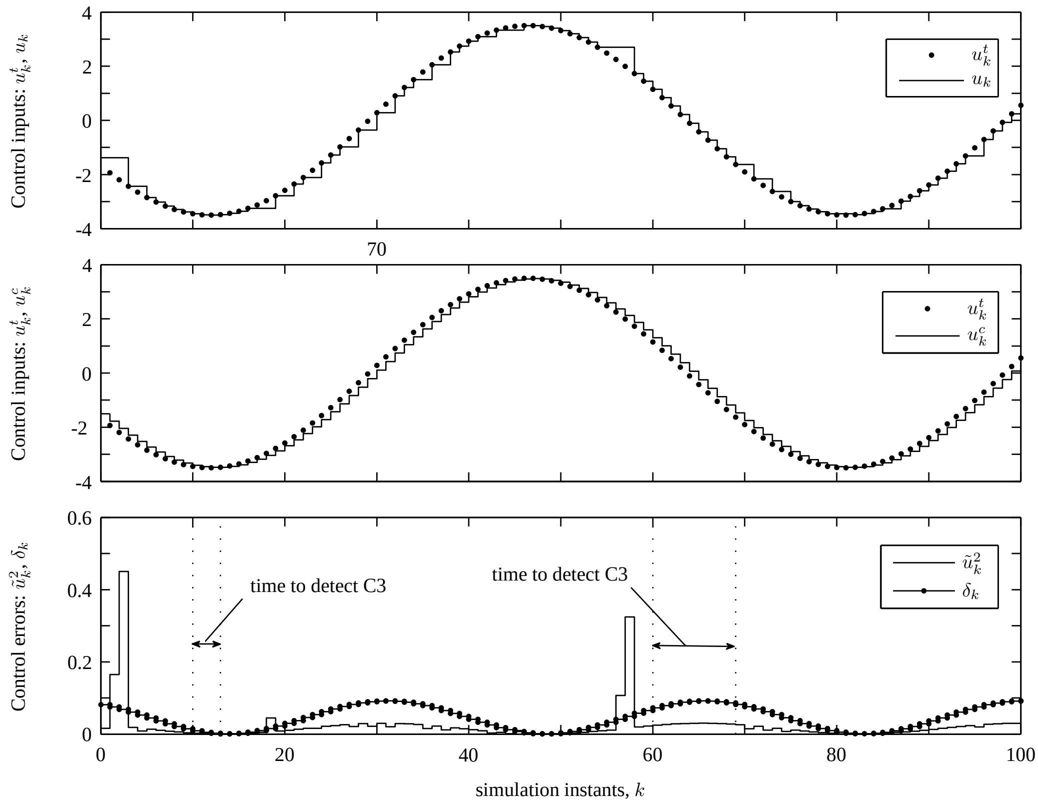

Figure 5. Control inputs involved in the NCS (control input applied in the plant u_k, transmitted control input u_k^t and control input used in the observer u_k^c), squared control error $\tilde{u}_k^2$ and its statistic δ_k in the simulation of Example 4.1.

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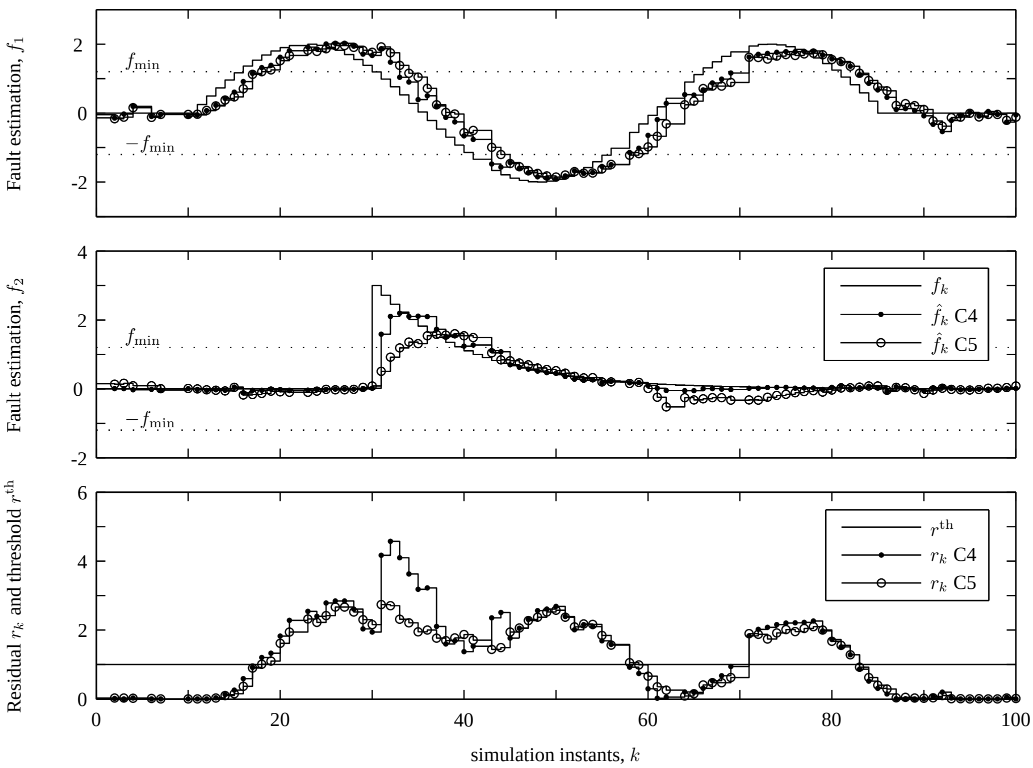

Figure 6. Fault estimation and fault detection performances in simulation for cases C4 and C5 in Table II. Markers $\bullet$ and $\circ$ denote instants when a measurement is received.

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